

ARE NUMBER FIELDS DETERMINED BY ARTIN L -FUNCTIONS?

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ABSTRACT. Let k be a number field, K/k a finite Galois extension with Galois group G , χ a faithful character of G . We prove that the Artin L -function $L(s, \chi, K/k)$ determines the Galois closure of K over \mathbb{Q} . In the special case $k = \mathbb{Q}$ it also determines the character χ .

Key words: Number fields, Galois extension, Artin L -function

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1. INTRODUCTION

Let k be a number field, K/k a finite Galois extension with Galois group G , χ a faithful character of G . In Theorem 6 we prove that the Artin L -function $L(s, \chi, K/k)$ determines the Galois closure \tilde{K} of K over \mathbb{Q} . In the special case $k = \mathbb{Q}$ we prove in Theorem 5 that the Artin L -function determines K and the (faithful) character χ . We give examples that in the case $k \neq \mathbb{Q}$ we cannot expect more, especially there exist non-isomorphic arithmetically equivalent fields which cannot be distinguished by Artin L -functions.

The restriction to faithful characters is natural: let K/k be a finite normal extension with $\text{Gal}(K/k) = G$, and let χ be a character of G with $\text{Ker}(\chi) \neq \{1\}$. Let F be the fixed field of $\text{Ker}(\chi)$, $H := G/\text{Ker}(\chi)$ the Galois group of F/k , $\varphi : H \rightarrow \mathbb{C}$, $\varphi(\sigma \text{Ker}(\chi)) := \chi(\sigma)$ for $\sigma \in G$. We have that

$$L(s, \chi, K/k) = L(s, \varphi, F/k),$$

and φ is faithful.

As particular cases we obtain that the Dedekind zeta function of a number field determines its normal closure ([4], Theorem 1, p. 345) and that a Galois number field is determined by any Artin L -function corresponding to a character which contains all irreducible characters of the Galois group, the result of [3].

2. PROPERTIES OF ARTIN L -FUNCTIONS

We do not give the definition of Artin L -functions, but we recall some fundamental properties of Artin L -functions needed in the sequel. Note that Artin L -functions are generalizations of Dedekind zeta functions ζ_K via

$$L(s, 1, K/K) = \zeta_K(s),$$

where K is a number field and 1 is the trivial character of the trivial group $\text{Gal}(K/K)$. We get further possibilities to write a Dedekind zeta function as a Artin L -function by using Propositions 1 and 2.

Proposition 1. *Let k be a number field, K/k a finite Galois extension with Galois group G , χ a character of G . Let N be a finite Galois extension of k which contains K , $U := \text{Gal}(N/k)$, $V := \text{Gal}(N/K)$. We identify the groups G and U/V . Let*

$$\tilde{\chi} : U \rightarrow \mathbb{C}, \quad \tilde{\chi}(\sigma) := \chi(\sigma V).$$

Then we have

$$L(s, \tilde{\chi}, N/k) = L(s, \chi, K/k).$$

Proof. This follows straightforward from the definition of L -functions: [1], p. 297, formula (8). \square

Proposition 2. *Let k be a number field, K/k a finite Galois extension with Galois group G . Let $k \subseteq F \subseteq K$ be an intermediate field, $H := \text{Gal}(K/F)$, χ a character of H , and χ^G be the induced character on G . Then we have*

$$L(s, \chi^G, K/k) = L(s, \chi, K/F).$$

Proof. This is a deep property proved by Emil Artin: [1], p. 297, formula (9). \square

Proposition 3. *Let K/\mathbb{Q} be a finite Galois extension with Galois group G , χ and φ characters of G . If*

$$L(s, \chi, K/\mathbb{Q}) = L(s, \varphi, K/\mathbb{Q})$$

then

$$\chi = \varphi.$$

Proof. This follows from the definition of Artin L -functions and Tschebotarev's density theorem. Alternately, it was proved in [2], p. 179, Theorem 1 that if $\chi \neq \varphi$ then the functions $L(s, \chi, K/\mathbb{Q})$ and $L(s, \varphi, K/\mathbb{Q})$ are linearly independent over \mathbb{C} . \square

We remark that Proposition 3 is not true for Artin L -functions in Galois extensions K/k with base $k \neq \mathbb{Q}$. E.g. let K/\mathbb{Q} be a normal S_3 -extension and denote by k the unique quadratic subfield of K/\mathbb{Q} . Then $\text{Gal}(K/k) = C_3$, the cyclic group of order 3. Denote by χ_2 and χ_3 the two non-trivial characters of C_3 . Then by Proposition 2 we get for $j = 2, 3$:

$$L(s, \chi_j^{S_3}, K/\mathbb{Q}) = L(s, \chi_j, K/k).$$

Using $\chi_2^{S_3} = \chi_3^{S_3}$ we get that $L(s, \chi_2, K/k) = L(s, \chi_3, K/k)$, but $\chi_2 \neq \chi_3$.

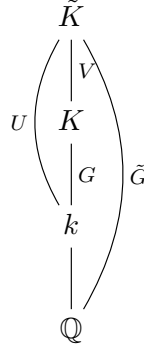
3. RESULTS

Proposition 4. *Let k be a number field, K/k a finite Galois extension with Galois group G , and χ be a faithful character of G . Let \tilde{K} be the Galois closure of K over \mathbb{Q} , \tilde{G} the Galois group of \tilde{K}/\mathbb{Q} , $U := \text{Gal}(\tilde{K}/k)$, and $V := \text{Gal}(\tilde{K}/K)$. We identify the groups G and U/V . Let*

$$\tilde{\chi} : U \rightarrow \mathbb{C}, \quad \tilde{\chi}(\sigma) := \chi(\sigma V),$$

and let $\tilde{\chi}^{\tilde{G}}$ be the induced character of $\tilde{\chi}$ on \tilde{G} . Then $\tilde{\chi}^{\tilde{G}}$ is faithful, and

$$L(s, \chi, K/k) = L(s, \tilde{\chi}^{\tilde{G}}, \tilde{K}/\mathbb{Q}).$$



Proof. We have

$$\text{Ker}(\tilde{\chi}) = V,$$

since χ is faithful. We have

$$\bigcap_{\sigma \in \tilde{G}} \sigma V \sigma^{-1} = 1,$$

since \tilde{K} is the Galois closure of K over \mathbb{Q} . We have

$$\text{Ker}(\tilde{\chi}^{\tilde{G}}) = \bigcap_{\sigma \in \tilde{G}} \sigma \text{Ker}(\tilde{\chi}) \sigma^{-1} = \bigcap_{\sigma \in \tilde{G}} \sigma V \sigma^{-1} = 1,$$

hence $\tilde{\chi}^{\tilde{G}}$ is faithful. We have

$$L(s, \chi, K/k) = L(s, \tilde{\chi}, \tilde{K}/k) = L(s, \tilde{\chi}^{\tilde{G}}, \tilde{K}/\mathbb{Q}),$$

by Propositions 1 and 2. □

We prove that a Galois number field is determined by any Artin L -function corresponding to a faithful character of the Galois group.

Theorem 5. *Let K_1/\mathbb{Q} , K_2/\mathbb{Q} be finite Galois extensions. For $j = 1, 2$ let G_j be the Galois group of K_j/\mathbb{Q} , χ_j a faithful character of G_j . If*

$$L(s, \chi_1, K_1/\mathbb{Q}) = L(s, \chi_2, K_2/\mathbb{Q})$$

then

$$K_1 = K_2 \text{ and } \chi_1 = \chi_2.$$

Proof. Let $N := K_1 \cdot K_2$. For $j = 1, 2$ let

$$\tilde{\chi}_j : \text{Gal}(N/\mathbb{Q}) \rightarrow \mathbb{C}, \quad \tilde{\chi}_j(\sigma) := \chi_j(\sigma \text{Gal}(N/K_j)),$$

where we identify G_j with the factor group $\text{Gal}(N/\mathbb{Q})/\text{Gal}(N/K_j)$. Then $\tilde{\chi}_j$ is a character of $\text{Gal}(N/\mathbb{Q})$, and

$$\text{Ker}(\tilde{\chi}_j) = \text{Gal}(N/K_j),$$

since χ_j is faithful. We have

$$L(s, \chi_j, K_j/\mathbb{Q}) = L(s, \tilde{\chi}_j, N/\mathbb{Q}),$$

by Proposition 1. If $L(s, \chi_1, K_1/\mathbb{Q}) = L(s, \chi_2, K_2/\mathbb{Q})$ then

$$L(s, \tilde{\chi}_1, N/\mathbb{Q}) = L(s, \tilde{\chi}_2, N/\mathbb{Q}),$$

hence, by Proposition 3,

$$\tilde{\chi}_1 = \tilde{\chi}_2.$$

This implies

$$\text{Gal}(N/K_1) = \text{Ker}(\tilde{\chi}_1) = \text{Ker}(\tilde{\chi}_2) = \text{Gal}(N/K_2),$$

hence

$$K_1 = K_2$$

and, by Proposition 3,

$$\chi_1 = \chi_2.$$

□

In the final example of this paper we show that we cannot expect a similar result to Theorem 5 for normal extensions K_j/k for $k \neq \mathbb{Q}$ and $j = 1, 2$. In the next theorem we give a version for relative extensions.

Theorem 6. *Let k_1 and k_2 be number fields. For $j = 1, 2$ let K_j/k_j be a finite Galois extension with the Galois group G_j , \tilde{K}_j the normal closure of K_j over \mathbb{Q} , χ_j a faithful character of G_j , $U_j := \text{Gal}(\tilde{K}_j/k_j)$, $V_j := \text{Gal}(\tilde{K}_j/K_j)$. We identify the groups G_j and U_j/V_j . Let*

$$\tilde{\chi}_j : U_j \rightarrow \mathbb{C}, \quad \tilde{\chi}_j(\sigma) := \chi_j(\sigma V_j).$$

Let $\tilde{G}_j := \text{Gal}(\tilde{K}_j/\mathbb{Q})$, and let $\tilde{\chi}_j^{\tilde{G}_j}$ be the induced character of $\tilde{\chi}_j$ on \tilde{G}_j . If

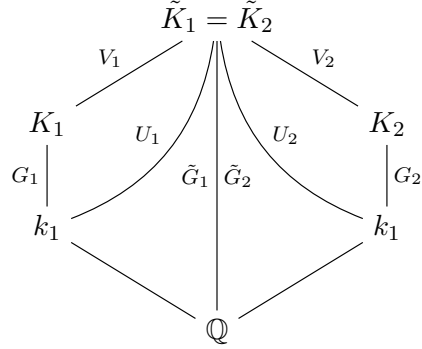
$$L(s, \chi_1, K_1/k_1) = L(s, \chi_2, K_2/k_2)$$

then

$$\tilde{K}_1 = \tilde{K}_2, \quad \tilde{G}_1 = \tilde{G}_2$$

and

$$\tilde{\chi}_1^{\tilde{G}_1} = \tilde{\chi}_2^{\tilde{G}_2}.$$



Proof. For $j = 1, 2$ the character $\tilde{\chi}_j^{\tilde{G}_j}$ is faithful and we have

$$L(s, \chi_j, K_j/k_j) = L(s, \tilde{\chi}_j^{\tilde{G}_j}, \tilde{K}_j/\mathbb{Q}),$$

by Proposition 4. If

$$L(s, \chi_1, K_1/k_1) = L(s, \chi_2, K_2/k_2)$$

then

$$L(s, \tilde{\chi}_1^{\tilde{G}_1}, \tilde{K}_1/\mathbb{Q}) = L(s, \tilde{\chi}_2^{\tilde{G}_2}, \tilde{K}_2/\mathbb{Q}),$$

hence

$$\tilde{K}_1 = \tilde{K}_2, \tilde{G}_1 = \tilde{G}_2$$

and

$$\tilde{\chi}_1^{\tilde{G}_1} = \tilde{\chi}_2^{\tilde{G}_2},$$

by Theorem 5. □

We obtain now the well-known result ([4], Theorem 1, p. 345) that the zeta function of an algebraic number field determines its normal closure.

Corollary 7. *Let K_1 and K_2 be number fields. For $j = 1, 2$ let \tilde{K}_j be the normal closure of K_j over \mathbb{Q} , $V_j := \text{Gal}(\tilde{K}_j/K_j)$, $\tilde{G}_j := \text{Gal}(\tilde{K}_j/\mathbb{Q})$, and let 1_{V_j} be the trivial character of V_j . If*

$$\zeta_{K_1}(s) = \zeta_{K_2}(s)$$

then

$$\tilde{K}_1 = \tilde{K}_2, \tilde{G}_1 = \tilde{G}_2$$

and

$$1_{V_1}^{\tilde{G}_1} = 1_{V_2}^{\tilde{G}_2}.$$

Proof. For $j = 1, 2$ let $k_j = K_j$, $G_j = \{1\}$, $\chi_j := 1_{G_j}$, $U_j := V_j$. The character χ_j is faithful, and

$$L(s, \chi_j, K_j/k_j) = \zeta_{K_j}(s).$$

We have

$$\tilde{\chi}_j : U_j \rightarrow \mathbb{C}, \tilde{\chi}_j(\sigma) := \chi_j(\sigma V_j) = \chi_j(1) = 1,$$

hence

$$\tilde{\chi}_j = 1_{V_j}.$$

If

$$\zeta_{K_1}(s) = \zeta_{K_2}(s)$$

then

$$L(s, \chi_1, K_1/k_1) = L(s, \chi_2, K_2/k_2),$$

hence

$$\tilde{K}_1 = \tilde{K}_2, \tilde{G}_1 = \tilde{G}_2$$

and

$$1_{V_1}^{\tilde{G}_1} = 1_{V_2}^{\tilde{G}_2},$$

by Theorem 6. □

We finish with an example which shows that all Artin L -functions in the Galois closures of the corresponding fields coincide. Let $k = k_1 = k_2 := \mathbb{Q}(\sqrt[4]{3})$, $K_1 := \mathbb{Q}(\sqrt[8]{3})$, $K_2 := \mathbb{Q}(\sqrt[8]{16 \cdot 3})$. The extensions K_1/k and K_2/k are Galois with Galois groups G_1 and G_2 of order 2. The fields K_1 and K_2 are non-isomorphic and have the same zeta function ([4], p. 350):

$$L(s, \text{Reg}_{G_1}, K_1/k) = \zeta_{K_1}(s) = \zeta_{K_2}(s) = L(s, \text{Reg}_{G_2}, K_2/k).$$

For $j = 1, 2$ we have

$$\text{Reg}_{G_j} = 1_{G_j} + \chi_j,$$

where χ_j is the non-trivial irreducible character of G_j . We have

$$L(s, 1_{G_1}, K_1/k) = \zeta_k(s) = L(s, 1_{G_2}, K_2/k),$$

hence

$$L(s, \chi_1, K_1/k) = L(s, \chi_2, K_2/k).$$

It follows that the Artin L -functions of K_1/k are identical with the Artin L -functions of K_2/k . We have

$$\tilde{K}_1 = \tilde{K}_2, \tilde{G}_1 = \tilde{G}_2 = G,$$

by Theorem 6, since Reg_{G_1} and Reg_{G_2} are faithful. Using the notation of Theorem 6, there is an outer automorphism α of G such that

$$V_2 = \alpha(V_1).$$

The Artin L -functions of \tilde{K}_1/K_1 are identical with the Artin L -functions of \tilde{K}_2/K_2 . This shows that K_1 and K_2 cannot be distinguished by Artin L -functions of \tilde{K}_1/K_1 and \tilde{K}_2/K_2 , nor by Artin L -functions of K_1/k and K_2/k .

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